Fuzzy Actions

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Abstract

This paper generalizes (fuzzifies) actions of a monoid or group on a set to deal with situations where imprecision and uncertainty are present. Fuzzy actions can handle the granularity of a set or even create it by defining a fuzzy equivalence relation on it.

Keywords: fuzzy action, fuzzy mapping, T-indistinguishability operator, fuzzy subgroup.

1 Introduction

Actions of a group or monoid G on a set I are a very useful tool in many branches of Mathematics and Computer Science. Paradigmatic examples are the actions of subgroups of the general linear group $\mathrm{GL}(n, \mathbb{R})$ on the vector space $\mathbb{R}^n$, the action of the projective linear group $\mathrm{PGL}(n+1, \mathbb{R})$ on the projective space $\mathbb{P}^n(\mathbb{R})$ and the action of the symmetry groups of regular polygons, friezes and wallpapers [10]. These examples are of geometric nature and are special cases of the concept of Geometry introduced by Klein in his Erlangen Program [6], where a Geometry is defined as a set and a group acting on it.

Actions are also useful in abstract algebra. In group theory, for example, translation and conjugation of a group on itself help prove important results such as the Lagrange Theorem [8].

In Computer Science, actions appear in Automata Theory and in Pattern Recognition, especially in character recognition problems [4] [11], where an image x can be compared with a prototype p by searching for an element g of a specific group such that the action of g on x transforms it into p ($gx = p$).

There are situations where imprecision, lack of accuracy or noise have to be taken into account or must be added to the problems to be solved. In the last example of character recognition, for instance, it is unlikely that we could find g of a "reasonable" group acting on the set of images or characters in such a way that gx is exactly p. In fact we expect to say that x corresponds to the character p when we can find g with gx only close or similar to p. In these cases, the action must be relaxed and allow it to consider imprecision and inaccuracy.

In this paper we present and develop the concept of fuzzy action that generalizes the idea of action of a group or monoid G on a set I. The action of an element $g \in G$ on an element $x \in I$ is not a precise element of I, but a fuzzy set encapsulating the imprecision given by the granularity of the system. According to Zadeh, granularity is one of the basic concepts that underlie human cognition [18] and the elements within a granule 'have to be dealt with as a whole rather than individually' [17].

> Informally, granulation of an object A results in a collection of granules of A, with a granule being a clump of objects (or points) which are drawn together by indistinguishability, similarity, proximity or functionality [18].

In fact, it will be proved that a fuzzy action α on a set I generates a fuzzy equivalence relation (an indistinguishability operator) E_I on I in a natural way and that from a crisp action on I and an indistinguishability operator on I satisfying an invariant condition with respect to the action (see Definition 3.19) a fuzzy action derives.

There are a few number of attempts to fuzzify actions on sets previous to this paper. Haddadi [5] and Roventa and Spircu [16] study fuzzy actions of fuzzy submonoids and fuzzy subgroups from an algebraic point of view. Lizasoain and Moreno [9] generalize results of [4] and [11] for the comparison of deformed images. In these papers the approach is rather different to the one proposed here.

The paper is organized as follows: After this introductory section, a section of preliminaries contains the basic definitions and properties of fuzzy subgroups, indistinguishability operators and fuzzy mappings needed on the paper. The composition of two mappings differs from the usual one in the sense that compatibility with the intermediate indistinguishability operator is required (cf. Definition 2.9). Consequently, subsequent properties of Section 2 are new. Section 3 contains the main results of the paper. The first subsection of Section 4 generalizes fuzzy actions of a group or monoid to fuzzy actions of fuzzy groups or monoids and specializes it to the restriction of a crisp action of G to a fuzzy subgroup or submonoid of G. The second subsection of Section 4 contains a couple of examples.

2 Preliminaries

This section contains some definitions and properties related to fuzzy subgroups and T-indistinguishability operators that will be needed later. Some definitions and properties of fuzzy maps needed in the paper will be stated.

Throughout the paper T will denote a given t-norm.

Fuzzy subgroups were introduced by Rosenfeld [15] as a natural generalization of the concept of subgroup and have been widely studied [12].

Definition 2.1. *Let G be a group and μ a fuzzy subset of X. μ is a T-fuzzy subgroup of G if and only if $T(\mu(g), \mu(h)) \leq \mu(gh^{-1})$ $\forall g, h \in G$.*

Proposition 2.2. *Let G be a group, e its identity element and μ a fuzzy subset of G such that $\mu(e) = 1$. Then μ is a T-fuzzy subgroup of G if and only if $\forall g, h \in G$ the following properties hold*

1. $\mu(g) = \mu(g^{-1})$
2. $T(\mu(g), \mu(h)) \leq \mu(gh)$.

Proof.

$\Rightarrow$)

If μ is a T-fuzzy subgroup with $\mu(e) = 1$, then

$$\mu(g) = T(\mu(e), \mu(g)) \leq \mu(g^{-1}).$$

By symmetry, $\mu(g) = \mu(g^{-1})$ holds.

Also $T(\mu(g), \mu(h)) = T(\mu(g), \mu(h^{-1})) \leq \mu(gh)$.

$\Leftarrow$)

$$T(\mu(g), \mu(h)) = T(\mu(g), \mu(h^{-1})) \leq \mu(gh^{-1}).$$

□

Proposition 2.3. *Let G be a group, e its identity element and μ a fuzzy subgroup of G such that $\mu(e) = 1$. Then the core H of μ (i.e.: the set of elements g of G such that $\mu(g) = 1$) is a (crisp) subgroup of G.*

Proof. Let $g, h \in H$.

$$1 = T(\mu(g), \mu(h)) = T(\mu(g), \mu(h^{-1})) \leq \mu(gh^{-1}).$$

and therefore $gh^{-1} \in H$. □

Definition 2.4. *A fuzzy relation E on a set X is a T-indistinguishability operator on X if and only for all x, y, z of X satisfies the following properties*

- $E(x, x) = 1$ *(Reflexivity)*
- $E(x, y) = E(y, x)$ *(Symmetry)*
- $T(E(x, y), E(y, z)) \leq E(x, z)$ *(Transitivity)*

T-indistinguishability operators extend the concept of equivalence relation and equality to the fuzzy framework and they are also called fuzzy equivalence and fuzzy equality relations. $E(x, y)$ can be viewed as the degree of similarity or indistinguishability between x and y. A general panorama on T-indistinguishability operators can be found in [14].

We recall the sup $-T$ product between fuzzy relations that will be needed in the study of fuzzy mappings.

Definition 2.5. *Let X, Y, Z be sets and $R : X \times Y \to [0,1]$ and $S : Y \times Z \to [0,1]$ fuzzy relations. The* $\sup -T$ *product $R \circ_T S$ of R and S is the fuzzy relation $R \circ_T S : X \times Z \to [0,1]$ defined for all $x \in X$, $z \in Z$ by*

$$R \circ_T S(x,z) = \sup_{y \in Y} T(R(x,y), S(y,z)).$$

Fuzzy mappings generalize the concept of mapping between two sets X and Y. The sets are supposed to be endowed with T-indistinguishability operators and compatibility of the fuzzy mappings with them is imposed. Interesting properties of fuzzy mappings can be found in [2] [3].

Definition 2.6. *Let E_X and E_Y be T-indistinguishability operators on two sets X and Y respectively. $R : X \times Y \to [0,1]$ is a fuzzy mapping from X onto Y if and only if for all $x, x' \in X$ and for all $y, y' \in Y$*

- $T(R(x,y), E_X(x,x'), E_Y(y,y')) \leq R(x',y')$
- $T(R(x,y), R(x,y')) \leq E_Y(y,y')$.

R is perfect if and only if

- *For all $x \in X$ there exists $y \in Y$ such that $R(x,y) = 1$.*

Definition 2.7. *A fuzzy mapping R from X onto Y is injective if and only if for all $x, x' \in X$ and for all $y, y' \in Y$*

$$T(R(x,y), R(x',y'), E_Y(y,y')) \leq E_X(x,x').$$

Definition 2.8. *Given a fuzzy mapping R from X onto Y, The degree* $\mathrm{Im}(y)$ *in which $y \in Y$ is in the image of R (Im(R)) is*

$$\mathrm{Im}(y) = \sup_{x \in X, y' \in Y} T(R(x,y'), E_Y(y',y)) = \sup_{x \in X} (R \circ_T E_Y(x,y)).$$

The infimum of the last expression for $y \in Y$, $\inf_{y \in Y}\{\mathrm{Im}(y)\}$, is the degree of surjectivity of R.

R is strong surjective if and only if its degree of surjectivity is 1.

Definition 2.9. *Let $R : X \times Y \to [0,1]$ and $S : Y \times Z \to [0,1]$ be fuzzy mappings. The composition of R and S is the fuzzy mapping $M = R \circ S : X \times Z \to [0,1]$ defined for all $x \in X$, $z \in Z$ by*

$$M(x,z) = (R \circ_T E_Y \circ_T S)(x,z) = \sup_{y, y' \in Y} T(R(x,y), E_Y(y,y'), S(y',z)).$$

N.B. This definition of composition of fuzzy maps is different from the usual one (please compare $R \circ S$ with $R \circ_T S$) as it considers the T-indistinguishability E_Y on Y.

Proposition 2.10. *If the composition mapping $M = R \circ S$ is injective and S is a perfect mapping, then R is injective.*

Proof. Injectivity of M means

$$E_X(x, x') \geq T(M(x, z), M(x', z'), E_Z(z, z'))$$

for all $x, x', z, z' \in Z$. In particular, for $z' = z$ we get

$$\begin{aligned} & E_X(x, x') \\ & \quad \geq \sup_{y,y',y'',y''' \in Y} T(R(x, y), E_Y(y, y'), S(y', z), R(x', y''), E_Y(y'', y'''), S(y''', z), E_Z(z, z)) \\ & \quad \geq \sup_{y,y' \in Y} T(R(x, y), E_Y(y, y'), S(y', z), R(x', y'), E_Y(y', y'), S(y', z)) \end{aligned}$$

the last inequality following by considering $y' = y'' = y'''$. Now, since S is perfect, for y' there exists $z_{y'}$ with $S(y', z_{y'}) = 1$ and the last expression is greater than or equal to

$$T(R(x, y), R(x', y'), E_Y(y, y'))$$

which means injectivity of R. $\square$

Definition 2.11. *Let $R : X \times Y \to [0, 1]$ and $S : Y \times X \to [0, 1]$ be fuzzy mappings. R is the inverse of S (and vice versa) if and only if*

- $T(R(x, y), E_Y(y, y'), S(y', x')) \leq E_X(x, x')$
- $T(S(y, x), E_X(x, x'), R(x', y')) \leq E_Y(y, y')$.

In other words, R is the inverse of S if and only if both compositions are respectively smaller than or equal to E_X and E_Y.

Definition 2.12. *Let R be a fuzzy relation $R : X \times Y \to [0, 1]$. The inverse relation S of R (usually denoted by R^{-1}) is the fuzzy relation $S : Y \times X \to [0, 1]$ defined for all $x \in X$, $y \in Y$ by $S(y, x) = R(x, y)$.*

Proposition 2.13. *If $R : X \times Y \to [0, 1]$ and $R^{-1} : Y \times X \to [0, 1]$ are both fuzzy mappings, then* *a) R is the inverse mapping of R^{-1}.*

b) R and R^{-1} are injective.

Proof.

a)

$$\begin{aligned} & T(R(x,y), E_Y(y,y'), R^{-1}(y',x')) \\ = \; & T(R(x,y), E_Y(y,y'), R(x',y')) \\ \leq \; & E_X(x,x') \end{aligned}$$

$$\begin{aligned} & T(R^{-1}(y,x), E_X(x,x'), R(x',y')) \\ = \; & T(R^{-1}(y,x), E_X(x,x'), R(x',y')) \\ \leq \; & E_Y(y,y'). \end{aligned}$$

b)

$$\begin{aligned} & T(R(x,y), R(x',y'), E_Y(y,y')) \\ = \; & T(R(x,y), R^{-1}(y',x'), E_Y(y,y')) \\ \leq \; & E_X(x,x') \end{aligned}$$

$$\begin{aligned} & T(R^{-1}(y,x), R^{-1}(y',x'), E_X(x,x')) \\ = \; & T(R^{-1}(y,x), R(x',y'), E_X(x,x')) \\ \leq \; & E_Y(y,y'). \end{aligned}$$

□

3 Fuzzy Actions

An action of a group or monoid G on a set I gives a way to transform the elements if I by assigning to every $g \in G$ and $x \in I$ another element gx of I. In geometry it generalizes the concept of symmetry and this abstraction allows us to consider and apply geometrical ideas to general and more abstract frameworks. Here is the formal definition of the action of a monoid on a set [8].

Definition 3.1. *Let G be a monoid with neutral element e and I a non-empty set. $\alpha : G \times I \to I$ is an action of G on I if and only if for all $g, h \in G$, $x \in I$*

1. $(hg)x = h(gx)$
2. $ex = x$

where $\alpha(g, x)$ is denoted by gx.

There is a number of applications of actions of monoids on sets. Some of them are presented below.

Example 3.2.

- *In a monoid G, left multiplication $\alpha(g, h) = gh$ is an action of G on itself.*
- *In a group G, conjugation $\alpha(g, h) = ghg^{-1}$ is an action of G on itself.*
- *The symmetry group of a regular polygon acts on its vertices.*
- *The general linear group $\mathrm{GL}(n, \mathbb{R})$, special linear group $\mathrm{SL}(n, \mathbb{R})$, orthogonal group $\mathrm{O}(n, \mathbb{R})$, special orthogonal group $\mathrm{SO}(n, \mathbb{R})$ and symplectic group $\mathrm{Sp}(n, \mathbb{R})$ act on the vector space $\mathbb{R}^n$, the action being multiplication of the vectors of $\mathbb{R}^n$ by the corresponding matrices.*

There are situations where either there is no possibility to define an action in a precise way or due to the nature of the problem it is convenient to consider imprecise actions. To mention only one example, consider the case of character or letter recognition. In a first step we try to match the character x to be identified with a designed prototype p by acting on x with an element g of, for example, the general linear group $\mathrm{GL}(2, \mathbb{R})$. In an ideal case we could find $g \in \mathrm{GL}(2, \mathbb{R})$ with $gx = p$ but in general we will only be able to find $g \in \mathrm{GL}(2, \mathbb{R})$ with gx similar or close to p.

In other cases, there may be granularity on the set I that does not permit defining precise concepts such as crisp actions. This can happen for example by the presence of an indistinguishability operator on I.

In these cases, the fuzzification of the action of monoids on sets is needed.

Definition 3.3. *Let G be a monoid with neutral element e, I a non-empty set and T a t-norm. $\alpha : G \times I \times I \to [0, 1]$ is a T-fuzzy action (or simply a fuzzy action) of G on I if and only if for all $g, h \in G$, $x \in I$*

1a) $T(\alpha(hg, x, y), \alpha(g, x, z)) \leq \alpha(h, z, y)$

1b) $T(\alpha(g, x, z), \alpha(h, z, y)) \leq \alpha(hg, x, y)$

2. $\alpha(e, x, x) = 1$.

$\alpha(g, x, y)$ can be interpreted as the degree to which y is the image of x under the action of $g \in G$.

NB. Properties 1a) and 1b) fuzzify the conditions

$$(hg)x = y \Rightarrow h(gx) = y$$

and

$$h(gx) = y \Rightarrow (hg)x = y$$

respectively.

Definition 3.4.

- *A fuzzy action α is quasi-perfect if and only if for all $g \in G$, $x \in I$ there exists $y \in I$ such that $\alpha(g, x, y) = 1$.*
- *α is perfect if and only if the previous y is unique.*

Lemma 3.5. *Let α be a fuzzy action of G on X. If for $g \in G$ and $x, y, y' \in I$ $\alpha(g, x, y) = \alpha(g, x, y') = 1$, then $\alpha(e, y, y') = 1$.*

Proof. From 1a),

$$1 = T(\alpha(g, x, y), \alpha(g, x, y')) \leq \alpha(e, y, y').$$

□

Given a quasi-perfect fuzzy action α of G on I we can consider the (crisp) equivalence relation $\sim$ on I defined by $x \sim y$ if and only if $\alpha(e, x, y) = 1$. and the fuzzy action $\overline{\alpha}$ of G on $\overline{I} = I/\sim$ defined by $\overline{\alpha}(g, \overline{x}, \overline{y}) = \alpha(g, x, y)$.

Proposition 3.6. *$\sim$ is an equivalence relation, $\overline{\alpha}$ is well defined and is a fuzzy action of G on $\overline{I} = I/\sim$.*

Proof.

Reflexivity

$x \sim x$, because $\alpha(e,x,x) = 1$.

Symmetry

$$1 = \alpha(e,x,y) = T(\alpha(e,x,x), \alpha(e,x,y)) \leq \alpha(e,y,x)$$

by Property 1b).

Transitivity

f $x \sim y$ and $y \sim z$, then

$$1 = T(\alpha(e,x,y), \alpha(e,y,z)) \leq \alpha(e,x,z)$$

by Property 1b).

In order to prove that $\overline{\alpha}$ is well defined, we must show that if $\alpha(e,x,x') = 1$ and $\alpha(e,y,y') = 1$, then $\alpha(g,x,y) = \alpha(g,x',y')$.

$$\begin{aligned}
\alpha(g,x,y) &= T(\alpha(g,x,y), \alpha(e,x,x'), \alpha(e,y,y')) \\
&= T(\alpha(g,x,y), \alpha(e,x',x), \alpha(e,y,y')) \\
&\leq \alpha(g,x',y')
\end{aligned}$$

and the result follows by symmetry.

It is straightforward to prove that $\overline{\alpha}$ is a fuzzy action. □

This result allows us to restrict the study of quasi-perfect fuzzy actions to perfect ones.

Proposition 3.7. *If α is a perfect fuzzy action of G on I and, for $g \in G$ and $x \in I$, $y_x \in I$ is the unique element of I such that $\alpha(g,x,y_x) = 1$, then $gx = y_x$ is a crisp action.*

Proof. Straightforward. □

Reciprocally it will be shown in Proposition 3.24 how to fuzzify crisp actions to obtain perfect fuzzy ones.

From now on, we will assume that G is a group.

Lemma 3.8. *If α is a fuzzy action of G on I, $h \in G$ and $x, z \in I$, then $\alpha(h^{-1},x,z) = \alpha(h,z,x)$.*

Proof. From 1.a) in Definition 3.3,

$$T(\alpha(hh^{-1},x,y),\alpha(h^{-1},x,z)) \leq \alpha(h,z,y)$$

or

$$T(\alpha(e,x,y),\alpha(h^{-1},x,z)) \leq \alpha(h,z,y)$$

In particular, taking $y = x$,

$$T(\alpha(e,x,x),\alpha(h^{-1},x,z)) = \alpha(h^{-1},x,z) \leq \alpha(h,z,x)$$

and the result follows by symmetry. □

$\alpha(e,x,y)$ provides the degree to which y is the transformed of x by the identity element e of G. It will then measure the degree in which we can consider x and y as equivalent or indistinguishable objects and will reflect the granularity on I. In fact this relation is a T-indistinguishability operator as it will be proved in Proposition 3.10.

Definition 3.9. *Let α be a fuzzy action of G on I. E_I is the fuzzy relation on I defined for all $x,y \in I$ by $E_I(x,y) = \alpha(e,x,y)$.*

Proposition 3.10. *E_I is a T-indistinguishability operator.*

Proof.

- Reflexivity:
$$E_I(x,x) = \alpha(e,x,x) = 1.$$
- Symmetry:
$$E_I(x,y) = \alpha(e,x,y) = \alpha(e,y,x) = E_I(y,x).$$
- T-transitivity:
$$\begin{aligned} T(E_I(x,y),E_I(y,z)) &= T(\alpha(e,x,y),\alpha(e,y,z)) \\ &\leq \alpha(e,x,z) = E_I(x,z) \end{aligned}$$
the inequality following from 1.b) in Definition 3.3.

□

For a crisp action, fixing g, the map $f_g : I \to I$ defined by $f_g(x) = gx$ is a bijection. The fuzzification of this result is the next Proposition 3.12.

Definition 3.11. *Let $\alpha : G \times I \times I \to [0,1]$ be a fuzzy action. Fixing $g \in G$, the fuzzy relation $R_g : I \times I \to [0,1]$ is defined for all $x, y \in I$ by*

$$R_g(x,y) = \alpha(g,x,y).$$

Proposition 3.12. *The fuzzy relation R_g is an injective fuzzy mapping with respect to the T-indistinguishability operator E_I.*

Proof.

- $$\begin{aligned} & T(R_g(x,y), E_I(x,x'), E_I(y,y')) \\ = \ & T(\alpha(g,x,y), \alpha(e,x,x'), \alpha(e,y,y')) \\ = \ & T(\alpha(g,x,y), \alpha(e,x',x), \alpha(e,y,y')) \\ \leq \ & \alpha(g,x',y') = R_g(x',y'). \end{aligned}$$

- $$\begin{aligned} & T(R_g(x,y), R_g(x,y')) \\ = \ & T(\alpha(g,x,y), \alpha(g,x,y')) \\ = \ & T(\alpha(g^{-1},y,x), \alpha(g,x,y')) \\ \leq \ & \alpha(e,y,y') = E_I(y,y'). \end{aligned}$$

- Injectivity:

$$\begin{aligned} & T(R_g(x,y), R_g(x',y'), E_\alpha(y,y')) \\ = \ & T(\alpha(g,x,y), \alpha(g,x',y'), E_\alpha(y,y')) \\ = \ & T(\alpha(g,x,y), \alpha(g^{-1},y',x'), \alpha(e,y,y')) \\ \leq \ & T(\alpha(g,x,y'), \alpha(g^{-1},y',x') \\ \leq \ & \alpha(e,x,x') = E_I(x,x'). \end{aligned}$$

□

Proposition 3.13. *R_g and $R_{g^{-1}}$ are inverse mappings.*

Proof.

- $$\begin{aligned} & T(R_g(x,y), E_I(y,y'), R_{g^{-1}}(y',x')) \\ & \quad = \; T(\alpha(g,x,y), \alpha(e,y,y'), \alpha(g^{-1},y',x')) \\ & \quad \leq \; T(\alpha(g,x,y), \alpha(g^{-1},y,x')) \\ & \quad \leq \; \alpha(e,x,x') = E_I(x,x'). \end{aligned}$$

- $$\begin{aligned} & T(R_{g^{-1}}(y,x), E_I(x,x'), R_g(x',y')) \\ & \quad = \; T(\alpha(g^{-1},y,x), \alpha(e,x,x'), \alpha(g,x',y')) \\ & \quad \leq \; T(\alpha(g^{-1},y,x'), \alpha(g,x',y')) \\ & \quad \leq \; \alpha(e,y,y') = E_I(y,y'). \end{aligned}$$

□

In a (crisp) action, we intend to consider as equivalent the elements that are equal but for the action of an element $g \in G$, so that two elements x and y of I are considered equivalent if and only if there exists $g \in G$ such that $y = gx$. The next definition fuzzifies this idea.

Definition 3.14. *Given a fuzzy action α, we define the fuzzy relation E_α on I by $E_\alpha(x,y) = \sup_{g \in G} \alpha(g,x,y)$ for all $x, y \in I$.*

Proposition 3.15. *If T is a left continuous t-norm, then E_α is a T-indistinguishability operator on I.*

Proof.

- Reflexivity:

$$E_\alpha(x,x) = \sup_{g \in G} \alpha(g,x,x) \geq \alpha(e,x,x) = 1.$$

- Symmetry:

$$E_\alpha(y,x) = \sup_{g \in G} \alpha(g,y,x) = \sup_{g \in G} \alpha(g^{-1},x,y) = \sup_{g \in G} \alpha(g,x,y) = E_\alpha(x,y).$$

- T-transitivity:

$$\begin{aligned} T(E_\alpha(x,y),E_\alpha(y,z)) &= T(\sup_{g\in G}\alpha(g,x,y),\sup_{h\in G}\alpha(h,y,z)) \\ &= \sup_{g,h\in G} T(\alpha(g,x,y),\alpha(h,y,z)) \\ &\le \sup_{g,h\in G} \alpha(hg,x,z) \\ &= E_\alpha(x,z) \end{aligned}$$

  the inequality following from 1.b) in Definition 3.3.

□

Definition 3.16. *Fixing $x \in I$, the column μ_x of E_α (i.e. the fuzzy set $\mu_x(y) = E_\alpha(x,y)$) is the fuzzy orbit of x.*

The fuzzy orbit of $x \in I$ is therefore the fuzzy equivalence class of x with respect to E_α.

Proposition 3.17. *Let $\alpha : G\times I\times I \to [0,1]$ be a fuzzy action. The fuzzy relation R_g is an injective fuzzy mapping with respect to the T-indistinguishability operator E_α.*

Proof.

- $$\begin{aligned} &T(R_g(x,y),E_\alpha(x,x'),E_\alpha(y,y')) \\ &\quad\le T(\sup_{l\in G}\alpha(l,x,y),\sup_{h\in G}\alpha(h,x,x'),\sup_{k\in G}\alpha(k,y,y'))) \\ &\quad= T(E_\alpha(x,y),E_\alpha(x,x'),E_\alpha(y,y')) \\ &\quad\le E_\alpha(x',y'). \end{aligned}$$

- $$\begin{aligned} &T(R_g(x,y),R_g(x,y')) \\ &\quad= T(\alpha(g,x,y),\alpha(g,x,y')) \\ &\quad\le T(E_\alpha(x,x'),E_\alpha(x,y')) \\ &\quad\le E_\alpha(y,y'). \end{aligned}$$

- Injectivity:

$$\begin{aligned} &T(R_g(x,y), R_g(z,t), E_\alpha(y,t)) \\ &\quad = \; T(\alpha(g,x,y), \alpha(g,z,t), E_\alpha(y,t)) \\ &\quad \leq \; T(E_\alpha(x,y), E_\alpha(z,t), E_\alpha(y,t)) \\ &\quad \leq \; E_\alpha(x,z). \end{aligned}$$

□

Considering an action α on a set I, useful (fuzzy or crisp) relations on I should be compatible with it in the sense that they should be invariant under the effect of the action α.

Definition 3.18. *Let α be a crisp action on I. A fuzzy relation R on I is invariant under α if and only if*

$$R(x,y) = R(\alpha(g,x), \alpha(g,y))$$

for all $g \in G$, $x, y \in I$.

For fuzzy actions, the previous definition can be generalized as follows.

Definition 3.19. *Let α be a fuzzy action of G on I and R a fuzzy relation on I. R is invariant under α if and only if*

$$T(R(x,y), \alpha(g,x,x'), \alpha(g,y,y')) \leq R(x',y')$$

for all $g \in G$ $x, y, x', y' \in I$.

Proposition 3.20. *E_I is invariant under α.*

Proof.

$$\begin{aligned} &T(E_I(x,y), \alpha(g,x,x'), \alpha(g,y,y')) \\ &\quad = \; T(\alpha(e,x,y), \alpha(g,x,x'), \alpha(g,y,y')) \\ &\quad \leq \; T(\alpha(g,x,y'), \alpha(g,x,x')) \\ &\quad = \; T(\alpha(g^{-1},y',x), \alpha(g,x,x')) \\ &\quad \leq \; \alpha(e,y',x') = E_I(x',y'). \end{aligned}$$

□

Proposition 3.21. *E_I is the smallest T-indistinguishability operator on X invariant under α.*

Proof. If E is a T-indistinguishability operator on I invariant under α, then

$$T(E(x,y),\alpha(g,x,x'),\alpha(g,y,y')) \leq E(x',y')$$

In particular,

$$\begin{aligned} T(E(x,y),\alpha(e,x,x'),\alpha(e,y,y')) &\leq E(x',y') \\ T(E(x,y),E_I(x,x'),E_I(y,y')) &\leq E(x',y'). \end{aligned}$$

Putting $x = y = x'$,

$$T(E(x,x),E_I(x,x),E_I(x,y')) \leq E(x,y')$$

and hence $E_I(x,y') \leq E(x,y')$.

□

NB. In fact we have only used reflexivity of E in the proof, so that Proposition 3.21 states that E_I is smaller than or equal to any reflexive fuzzy relation of I invariant under α.

Proposition 3.22. *E_α is invariant under α.*

Proof.

$$\alpha(g,x,x') \leq E_\alpha(x,x') \quad \text{and} \quad \alpha(g,y,y') \leq E_\alpha(y,y')$$

and hence

$$\begin{aligned} &T(E_\alpha(x,y),\alpha(g,x,x'),\alpha(g,y,y')) \\ &\leq T(E_\alpha(x,y),E_\alpha(x,x'),E_\alpha(y,y')) \\ &\leq E_\alpha(x',y'). \end{aligned}$$

□

Corollary 3.23. $E_I \leq E_\alpha$.

In Proposition 3.7 we have proved that a perfect fuzzy action generates a crisp action in a natural way. Reciprocally, from a crisp fuzzy action and a T-indistinguishability operator invariant under this action a fuzzy action can be obtained in again a natural way.

Proposition 3.24. *Let E be a T-indistinguishability operator on a set X invariant under the (crisp) action $x \to gx$ of a group G on I. The mapping $\alpha : G \times I \times I \to [0,1]$ defined for all $g \in G$ and for all $x, y \in I$ by*

$$\alpha(g, x, y) = E(gx, y)$$

is a perfect fuzzy action of G on I. Moreover, $E(x, y) = \alpha(e, x, y) = E_I(x, y)$.

Proof.

- 1a)

$$\begin{aligned} T(\alpha(hg, x, y), \alpha(g, x, z)) &= T(E(hgx, y), E(gx, z)) \\ &= T(E(hgx, y), E(hgx, hz)) \\ &\leq E(hz, y) = \alpha(h, z, y). \end{aligned}$$

- 1b)

$$\begin{aligned} T(\alpha(g, x, z), \alpha(h, z, y)) &= T(E(gx, z), E(hz, y)) \\ &= T(E(hgx, hz), E(hz, y) \\ &\leq E(hgx, y) = \alpha(hg, x, y). \end{aligned}$$

- 2.
$$\alpha(e, x, x) = 1 = E(ex, x) = E(x, x) = 1.$$

- α is perfect.
$$\alpha(g, x, gx) = E(gx, gx) = 1.$$

- $E(x, y) = \alpha(e, x, y) = E_I(x, y)$.

□

Proposition 3.25. *Fixing $x \in I$, the fuzzy subset μ of G defined by*

$$\mu(g) = \alpha(g, x, x)$$

is a fuzzy subgroup of G with $\mu(e) = 1$, where e is the identity element of G.

Proof.

- $$T(\mu(g), \mu(h)) = T(\alpha(g, x, x), \alpha(h, x, x)) \leq \alpha(gh, x, x) = \mu(gh).$$

- $$\mu(g) = \alpha(g, x, x) = \alpha(g^{-1}, x, x) = \mu(g^{-1}).$$

- $$\mu(e) = \alpha(e, x, x) = 1.$$

□

Definition 3.26. *μ is the isotropy fuzzy subgroup of $x \in I$.*

Proposition 3.27. *If μ is the isotropy fuzzy subgroup of $x \in I$, then the fuzzy relation E_x on G defined for all $g, h \in G$ by*

$$E_x(g, h) = \mu(gh^{-1})$$

is a T-indistinguishability operator on G.

Proof. It is a particular case of Proposition 3.5 in [3]. □

Definition 3.28. *Let α be a fuzzy action of G on I. If the t-norm is left continuous, then the fuzzy subset μ of G defined by*

$$\mu(g) = \inf_{x \in I} \alpha(g, x, x)$$

is a fuzzy subgroup of G with $\mu(e) = 1$ where e is the identity element of G.

Proof.

- $$\begin{aligned} T(\mu(g), \mu(h)) &= T(\inf_{x \in I} \alpha(g, x, x), \inf_{y \in I} \alpha(h, y, y)) \\ &= \inf_{x, y \in I} T(\alpha(g, x, x), \alpha(h, y, y)) \\ &\leq \inf_{x \in I} T(\alpha(g, x, x), \alpha(h^{-1}, x, x)) \\ &\leq \inf_{x \in I} \alpha(gh^{-1}, x, x) = \mu(gh^{-1}). \end{aligned}$$

- $$\mu(e) = \inf_{x \in I} \alpha(e, x, x) = 1.$$

□

4 Concluding Remarks

The proposed definition of fuzzy action seems to be a good tool to consider the effect of a monoid or group on a set in settings where imprecision and uncertainty are present. In Subsection 4.1 we propose its generalization (restriction) to fuzzy actions of fuzzy subgroups and its specification when the action is crisp. In Subsection 4.2 a couple of examples show the potential of fuzzy actions.

4.1 Restriction of Fuzzy Actions to Fuzzy Subgroups

The definition of fuzzy action and the results of the previous section can be easily generalized or restricted to fuzzy subgroups.

Definition 4.1. *If $\alpha : G \times I \times I \to [0,1]$ is a fuzzy action of G on I and μ is a fuzzy subgroup of G with $\mu(e) = 1$ (e the identity element of G), then the restriction α_μ of α to μ is the mapping $\alpha_\mu : G \times I \times I \to [0,1]$ defined for all $g \in G$ and $x, y \in I$ by*

$$\alpha_\mu(g, x, y) = T(\alpha(g, x, y), \mu(g)).$$

α_μ satisfies most of the properties of the preceding section.

If the action is crisp, then we obtain a fuzzy action defined by a fuzzy subgroup. Definition 4.1 becomes then

Definition 4.2. *Let gx be a (crisp) action of G on I and μ a fuzzy subgroup of G with $\mu(e) = 1$ (e the identity element of G). The restriction of the action to μ is the fuzzy action $\alpha : G \times I \times I \to [0,1]$ defined by*

$$\alpha(g, x, y) = \begin{cases} \mu(g) & \text{if } gx = y \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Examples

The following two examples illustrate the use of fuzzy actions in two different contexts.

Example 4.3. *Consider the special orthogonal group* $\mathrm{S}(2,\mathbb{R})$ *acting on* $\mathbb{R}^2$*. The elements of* $\mathrm{S}(2,\mathbb{R})$ *can be represented by matrices* g_θ *of the form*

$$g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The action of g *on a vector* (x,y) *of* $\mathbb{R}^2$ *is*

$$g_\theta(x,y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

We can add imprecision to $\mathbb{R}^2$ *by considering, for example, the* T*-indistinguishability operator (*T *the Łukasiewicz t-norm) on* $\mathbb{R}^2$ *defined for all* $(x,y),(x',y') \in \mathbb{R}^2$ *by* $E((x,y),(x',y')) = \max(1-\sqrt{(x-x')^2+(y-y')^2}, 0)$*. From the action and* E *we can define the fuzzy action* α *of* $\mathrm{S}(2,\mathbb{R})$ *on* $\mathbb{R}^2$ *for all* $g_\theta \in \mathrm{S}(2,\mathbb{R})$ *and* $(x,y),(x',y') \in \mathbb{R}^2$ *by*

$$\begin{aligned}\alpha(g_\theta,(x,y),(x',y')) &= E(g_\theta(x,y),(x',y')) \\ &= \max(1-\sqrt{(x\cos\theta - y\sin\theta - x')^2 + (x\sin\theta + y\cos\theta - y')^2}, 0).\end{aligned}$$

Example 4.4. *A challenging and probably insolvable problem in Music Theory is the selection of a good scale* [1]*. Nowadays, there is a consensus to divide the scale in twelve semitones having equal ratios when tuning keyboards. This means that starting from a note of a given frequency* f*, the next note above in the scale will have frequency* $2^{\frac{1}{12}} \times f$ *while the frequency of the one below will be* $2^{-\frac{1}{12}} \times f$*. Given a melody* I*, by multiplying the frequency of all its notes by* $2^{\frac{k}{12}}$ *for a fixed* $k \in \mathbb{Z}$ *we will obtain the same melody transposed by* k *semitones. Transposition gives then a group action* $G \times I \to I$ $(g,x) \to gx$ *of the multiplicative group* $G = \{2^{\frac{k}{12}} \mid k \in \mathbb{Z}\}$ *on the melody* I*. Other instruments than keyboards may use a different scale and the transpositions performed by these instruments may not exactly coincide with its performance by a keyboard. So we must be flexible with the action of* G *by allowing imprecision. The* T*-indistinguishability operator* E *on* I *(*T *the Product t-norm) defined by* $E(x,y) = (\min(\frac{f_x}{f_y}, \frac{f_y}{f_x}))^{128}$ [1] *where* f_x *denotes the frequency of the note* $x \in I$ *is invariant under the action of* G *on* I *and can be used to fuzzify the previous action to* $\alpha(g,x,y) = E(gx,y)$ *of* G *on* I *(if* $g = 2^{\frac{k}{12}}$*, then* $\alpha(2^{\frac{k}{12}},x,y) = (\min(\frac{2^{\frac{k}{12}}\times f_x}{f_y}, \frac{f_y}{2^{\frac{k}{12}}\times f_x}))^{128}$*) and so introducing the needed imprecision.*

[1] The exponent is meant to fit the values of E with heuristics of the Just-noticeable difference of pitches in experimental psychology [7], [13].

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